The Radiation Properties of a Parallel-Plane Waveguide in a Transversely Magnetized, Homogeneous Plasma

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Summary—**The Wiener-Hopf technique is used to study the radiation from a parallel-plane waveguide embedded in a homogeneous anisotropic plasma in which the external magnetic field is perpendicular to the direction of propagation and parallel to the perfectly conducting planes of the guide. The incident field in the guide is a TEM wave, which propagates in the positive z direction. The parallel-plane guide terminates at $z=0$, causing a reflected field in the waveguide, a radiation field, and a surface wave that is guided along the outer surface of one of the perfect conductors. Expressions are found for these field components, and the results are discussed for the different frequency ranges.**

Introduction

One of the classical problems in electromagnetic theory is that of radiation from a parallel-plane waveguide or a circular waveguide into free space [1], [2], [3]. In this paper, we shall study the radiation from a parallel-plane waveguide which is embedded not in free space, but in a homogeneous anisotropic plasma. The geometry of the problem is sketched in Fig. 1; the incident field is a TEM wave propagating in the positive z direction. This wave is confined by two perfectly conducting planes of zero thickness separated by a distance $2a$ that terminate at $z=0$. The planes are embedded in a gyrotropic medium in which the external magnetic field lies in the positive y direction; *i.e.*, transverse to the direction of propagation of the incident wave and parallel to the perfectly conducting planes. Since there is no field variation in the y direction, the problem is two dimensional. The termination of the perfect conductors at $z=0$ causes a reflected field in the parallel-plane waveguide, a radiation field, and a surface wave propagating toward $z=-\infty$ which is guided by the outer surface of one of the perfect conductors. Expressions are found for the magnitudes of these field components by using the Wiener-Hopf technique. Seshadri [4], [5] has treated a similar problem with surface waves and a semi-infinite, perfectly conducting plane, and we shall make use of his notation and some of his results.

The Plasma Model

The medium surrounding the parallel-plane waveguide of Fig. 1 is a uniform plasma, and there is a uniform magnetic field B_0, impressed in the y direction. We shall make the following assumptions:

1) The plasma as a whole is at rest.
2) It is "temperate."
3) It is lossless.
4) The oscillations of the ions are negligible compared with those of the electrons.
5) The magnetic field of the waves in the plasma is much smaller than B_0.

Then the electric and magnetic fields satisfy Maxwell's equations

$$\nabla \times H = j\omega\epsilon_0 \underline{\underline{\mathcal{E}}} \cdot E \tag{1}$$

$$\nabla \times E = -j\omega\mu_0 H \tag{2}$$

where all fields have a $e^{j\omega t}$ time dependence. Here ϵ_0 and μ_0 are the dielectric constant and permeability of free space, and the relative dyadic dielectric constant $\underline{\underline{\mathcal{E}}}$ is

$$\underline{\underline{\mathcal{E}}} = \begin{vmatrix} \epsilon_1 & 0 & j\epsilon_2 \\ 0 & \epsilon_3 & 0 \\ -j\epsilon_2 & 0 & \epsilon_1 \end{vmatrix} \tag{3}$$

when the static magnetic field lies in the positive y direction. The components ϵ_1, ϵ_2, and ϵ_3 are given by

$$\epsilon_1 = \frac{\Omega^2 - R^2 - 1}{\Omega^2 - R^2} \tag{4}$$

$$\epsilon_2 = \frac{R}{\Omega(\Omega^2 - R^2)} \tag{5}$$

$$\epsilon_3 = 1 - \frac{1}{\Omega^2} \tag{6}$$

$$\Omega = \frac{\omega}{\omega_P} \tag{7}$$

$$R = \frac{\omega_c}{\omega_P} \tag{8}$$

$$\omega_p^{\ 2} = \frac{Ne^2}{m\epsilon_0} \tag{9}$$

$$\omega_c = -\frac{eB_0}{m}\,. \tag{10}$$

The plasma and gyromagnetic frequencies are ω_P and ω_c, respectively, e is the electron charge, m the electron mass, and N the average electron density.

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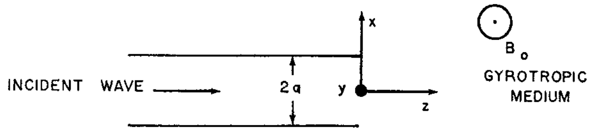

Fig. 1—Parallel-plane waveguide in a transversely magnetized homogeneous plasma.

For the two-dimensional situation of Fig. 1, there is only one component of magnetic field present H_y and only two electric field components, E_z and E_x. Using (1) and (2) one can show

$$E_x = \frac{j\epsilon_1}{\omega\epsilon_0\epsilon}\frac{\partial H_y}{\partial z} - \frac{\epsilon_2}{\omega\epsilon_0\epsilon}\frac{\partial H_y}{\partial x} \tag{11}$$

$$E_z = -\frac{\epsilon_2}{\omega\epsilon_0\epsilon}\frac{\partial H_y}{\partial z} - \frac{j\epsilon_1}{\omega\epsilon_0\epsilon}\frac{\partial H_y}{\partial x} \tag{12}$$

$$\left\{\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} + k^2\right\} H_y = 0 \tag{13}$$

where

$$\epsilon = \epsilon_1^2 - \epsilon_2^2 \tag{14}$$

$$k = k_0\sqrt{\frac{\epsilon}{\epsilon_1}} \tag{15}$$

$$k_0 = \omega\sqrt{\mu_0\epsilon_0}. \tag{16}$$

The Propagating Modes

There are two regions in which guided waves can exist when a parallel-plane waveguide is immersed in gyrotropic medium as in Fig. 1. First, surface waves can exist on the top or bottom of the perfectly conducting planes outside the waveguide; secondly, there are the modes inside the waveguide. Consider a single perfectly conducting plane immersed in a plasma as shown in Fig. 2. Seshadri has shown that the magnetic field of the surface wave for $x > 0$ is

$$H_y = H_s \exp\left[-\frac{k_0|\epsilon_2|}{\sqrt{\epsilon_1}}x \pm j\sqrt{\epsilon_1}\,k_0 z\right] \qquad x > 0, \quad \epsilon_2 \gtrless 0 \tag{17}$$

provided that $\epsilon_1 > 0$ [4]. For $\epsilon_2 >$ or $\epsilon_2 < 0$ the surface wave can travel only in the negative or positive z directions, respectively, *i.e.*, the wave has an undirectional character. A similar equation holds for $x < 0$.

The E waves which can exist in a parallel-plane waveguide filled with a gyrotropic medium ($|x| < a$ in Fig. 1) have been studied by Bers [6]. The lowest order mode is a TEM mode whose magnetic field is

$$H_y(x, z) = H_{y0} \exp\left[\pm\frac{k_0\epsilon_2 x}{\sqrt{\epsilon_1}} \mp jk_0\sqrt{\epsilon_1}\,z\right] \qquad -a \le x \le a. \tag{18}$$

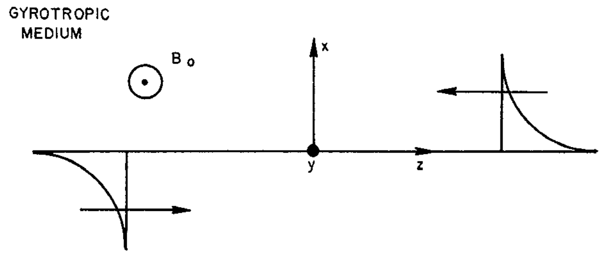

Fig. 2—Perfectly conducting plane in anisotropic plasma, $\epsilon_2 > 0$.

The upper and lower signs correspond to propagation in the positive and negative z directions, respectively.

The Formulation of the Problem

Having investigated the surface wave and the waveguide modes that can exist, we shall proceed with the boundary value problem. The total field H_{yT} may be written in terms of the incident and scattered fields.

$$H_{yT}(x, z) = H_y(x, z) + \exp\left[\frac{k_0\epsilon_2 x}{\sqrt{\epsilon_1}} - jk_0\sqrt{\epsilon_1}\,z\right] \qquad -a \le x \le a$$
$$= H_y(x, z), \qquad |x| > a. \tag{19}$$

The incident field is the term

$$\exp\left[\frac{k_0\epsilon_2 x}{\sqrt{\epsilon_1}} - jk_0\sqrt{\epsilon_1}\,z\right]$$

it represents a TEM wave of unit amplitude propagating in the region $-a < x < a$. The scattered field $H_y(x, z)$ consists of the reflected modes in the waveguide, the surface wave on the outside of the waveguide, and the radiation field.

Now let us consider the boundary conditions which must be satisfied. First, there can be no tangential electric field along the perfect conductor so that from (12)

$$\frac{\epsilon_2\partial H_y}{\partial z} + \frac{j\epsilon_1\partial H_y}{\partial x} = 0, \qquad x = \pm a, \quad z < 0. \tag{20}$$

Secondly, E_z and H_y are continuous at $x = \pm a$, $z > 0$; in fact, E_z is continuous for all z, if (20) is taken into consideration. In addition to the boundary conditions, there are the edge conditions which require that the components of current density normal to the edges vanish as $\rho^{1/2}$ and the components tangential vanish as $\rho^{-1/2}$, where ρ is the distance to the edge. $H_y(x, z)$ must, of course, satisfy the Helmholz equation (13).

Let $F(x, z)$ be the two-sided Laplace transform of $H_y(x, z)$,

$$F(x, s) = \int_{-\infty}^{\infty} H_y(x, z)e^{-sz}dz. \tag{21}$$

Then applying the transform to the Helmholz equation (13) gives

$$\left\{\frac{\partial^2}{\partial x^2} + s^2 + k^2\right\} F(x, s) = 0 \tag{22}$$

Because $F(x, s)$ is a solution of (22), it has the form

$$F(x,s) = A(s)e^{-jW(x-a)}, \qquad x \geq a \tag{23a}$$

$$= B(s)e^{-jWx} + C(s)e^{jWx}, \qquad -a \leq x \leq a \tag{23b}$$

$$= D(s)e^{jW(x+a)}, \qquad x \leq -a \tag{23c}$$

where the branch of W, $W=\sqrt{s^2+k^2}$ is chosen which equals k, if $s=0$. The positive exponential $\exp jW(x-a)$ is not a suitable solution for $F(x, s)$, $x>a$, since it becomes infinite as $x\rightarrow\infty$; similar reasoning excludes $\exp -jW(x+a)$ for $x<-a$.

We shall use the boundary conditions and the Wiener-Hopf technique to find $A(s)$, $B(s)$, $C(s)$, and $D(s)$; the inverse transform then yields the scattered field. By the continuity of H_{yT} at $x=a$, $z>0$

$$H_y(a+, z) = H_y(a-, z) + \left[\exp\frac{\epsilon_2 k_0 a}{\sqrt{\epsilon_1}} - jk_0\sqrt{\epsilon_1}\, z\right] \qquad z > 0 \tag{24}$$

or using inverse transforms

$$\frac{1}{2\pi j}\int_{c_0-j\infty}^{c_0+j\infty}\left\{A(x) - B(s)e^{-jWa} - C(s)e^{jWa} - \frac{\exp\dfrac{\epsilon_2 k_0 a}{\sqrt{\epsilon_1}}}{s+jk_0\sqrt{\epsilon_1}}\right\} e^{sz}ds = 0 \qquad z > 0. \tag{25}$$

When z is positive, the contour may be closed in a semicircle at infinity in the left half plane with no additional contribution from the semicircle. A suitable solution for (25) is obtained by equating the bracketed quantity inside the integral to some function $L_1(s)$ which is analytic for Real $(s)<$ Real (jk).

$$A(s) - B(s)e^{-jWa} - C(s)e^{jWa} = L_1(s) + \frac{\exp\left[\epsilon_2 k_0 a/\sqrt{\epsilon_1}\right]}{s+j\sqrt{\epsilon_1}\, k_0}\cdot \tag{26}$$

For convenience we shall assume that k has a small imaginary part which will be set equal to zero in the final result. Similarly, from the continuity of $H_y(x, z)$ at $x=-a$, $z>0$

$$D(s) - B(s)e^{jWa} - C(s)e^{-jWa} = L_2(s) + \frac{\exp - \epsilon_2 k_0 a/\sqrt{\epsilon_1}}{s+j\sqrt{\epsilon_1}\, k_0}\cdot \tag{27}$$

Since E_z is continuous at $x=a$, from (12),

$$\epsilon_2\frac{\partial H_y(a+, z)}{\partial z} + j\epsilon_1\frac{\partial H_y(a+, z)}{\partial x} = \epsilon_2\frac{\partial H_y(a-, z)}{\partial z} + j\epsilon_1\frac{\partial H_y(a-, z)}{\partial x} \qquad -\infty < z < \infty. \tag{28}$$

Taking the transform of (28) yields

$$[\epsilon_2 s + \epsilon_1 W]A(s) - [\epsilon_2 s + \epsilon_1 W]e^{-jWa}B(s) - [\epsilon_2 s - \epsilon_1 W]e^{jWa}C(s) = 0. \tag{29}$$

Similarly from the continuity of E_z at $x=-a$

$$[\epsilon_2 s - \epsilon_1 W]D(s) - [\epsilon_2 s + \epsilon_1 W]e^{jWa}B(s) - [\epsilon_2 s - \epsilon_1 W]e^{-jWa}C(s) = 0. \tag{30}$$

Because E_z vanishes on the perfect conductor,

$$\frac{1}{2\pi j}\int_{c_0-j\infty}^{c_0+j\infty}\left[\epsilon_2 s + j\epsilon_1\frac{\partial}{\partial x}\right]F(a+, s)e^{sz}ds = 0, \quad z < 0 \tag{31}$$

therefore,

$$[\epsilon_2 s + \epsilon_1 W]A(s) = R_1(s) \tag{32}$$

where $R_1(s)$ is analytic in Real $(s)>-$Real (jk). In a like manner

$$[\epsilon_2 s - \epsilon_1 W]D(s) = R_2(s). \tag{33}$$

Solving (26)–(30) for $A(s)$ and $D(s)$ in terms of $L_1(s)$ and $L_2(s)$, and using (32) and (33) yields

$$\frac{2\epsilon_1 We^{2jWa}}{\epsilon_1^2W^2 - \epsilon_2^2s^2}R_1(s) = L_1(s)e^{2jWa} - L_2(s) + \frac{1}{s+j\sqrt{\epsilon_1}\, k_0}\left\{\exp\left[\frac{\epsilon_2}{\sqrt{\epsilon_1}}k_0 a + 2jWa\right] - \exp\left[-\frac{\epsilon_2}{\sqrt{\epsilon_1}}k_0 a\right]\right\} \tag{34}$$

$$-\frac{2\epsilon_1 We^{2jWa}}{\epsilon_1^2W^2 - \epsilon_2^2s^2}R_2(s) = -L_1(s) + e^{2jWa}L_2(s) + \frac{1}{s+j\sqrt{\epsilon_1}\, k_0}\cdot\left\{-\exp\left[\frac{\epsilon_2}{\sqrt{\epsilon_1}}k_0 a\right] + \exp\left[-\frac{\epsilon_2}{\sqrt{\epsilon_1}}k_0 a + 2jWa\right]\right\}. \tag{35}$$

THE APPLICATION OF THE WIENER-HOPF TECHNIQUE

Eqs. (34) and (35) contain sufficient information for determining the scattered field. Adding and subtracting (34) and (35) gives

$$\frac{\epsilon_1[R_1(s) - R_2(s)]}{j\epsilon[s^2 + k_0^2\epsilon_1]e^{-jWa}\dfrac{\sin Wa}{W}} = L_1(s) + L_2(s) + \frac{2\cosh\left[\dfrac{\epsilon_1}{\sqrt{\epsilon_1}}k_0a\right]}{s + j\sqrt{\epsilon_1}k_0} \tag{36}$$

and

$$\frac{\epsilon_1 W[R_1(s) + R_2(s)]}{\epsilon[s^2 + k_0^2\epsilon_1]e^{-jWa}\cos Wa} = L_1(s) - L_2(s) + \frac{2\sinh\left[\dfrac{\epsilon_2}{\sqrt{\epsilon_1}}k_0a\right]}{s + j\sqrt{\epsilon_1}\,k_0}. \tag{37}$$

We shall solve each of these equations using the Wiener-Hopf technique, and then add the results to find $R_1(s)$ and $R_2(s)$. Consider (36) first. One can show (see [7]) that

$$\frac{e^{-jWa}\sin Wa}{W} = M(s) = M_+(s)M_-(s) \tag{38}$$

where

$$M_+(s) = \left[\frac{\sin ka}{k}\right]^{1/2}\exp\left[-\chi_1(s) - \frac{ja}{\pi}W\cos^{-1}\frac{s}{jk}\right] \prod_{n=1}^{\infty}\left(1 + \frac{s}{C_n}\right)e^{-s/C_n} \tag{39}$$

$$\chi_1(s) = s\sum_{n=1}^{\infty}\left[\frac{a}{n\pi} - \frac{1}{C_n}\right] + \frac{as}{\pi}\left[1 - C' + \ln\frac{2\pi}{jka}\right] \tag{40}$$

$$C_n = \sqrt{\left(\frac{n\pi}{a}\right)^2 - k^2} \tag{41}$$

$$C' = 0.5772 \text{ (Euler's constant)}. \tag{42}$$

Here $M_+(s)$ is analytic in Real $(s) >$ Real $(-jk)$ and $M_-(s)$ is analytic in Real $(s) <$ Real (jk). Because $M(s)$ is an even function of (s), $M_+(s) = M_-(-s)$, and as $|s| \to \infty$, $M_+(s)$ and $M_-(s)$ are asymptotic to $|s|^{-\frac{1}{2}}$ in the right and left half planes, respectively. Thus we may rewrite (36) as

$$\frac{\epsilon_1[R_1(s) - R_2(s)]}{j\epsilon[s + jk_0\sqrt{\epsilon_1}]M_+(s)} + \frac{4jk_0\sqrt{\epsilon_1}\cosh\dfrac{\epsilon_2}{\sqrt{\epsilon_1}}k_0aM_-(-jk_0\sqrt{\epsilon_1})}{s + jk_0 + \sqrt{\epsilon_1}}$$
$$= [s - jk_0\sqrt{\epsilon_1}]M_-(s)\left[L_1(s) + L_2(s) + \frac{2\cosh\dfrac{\epsilon_2}{\sqrt{\epsilon_1}}k_0a}{s + jk_0\sqrt{\epsilon_1}}\right] + \frac{4jk_0\sqrt{\epsilon_1}\cosh\dfrac{\epsilon_2}{\sqrt{\epsilon_1}}k_0aM_-(-jk_0\sqrt{\epsilon_1})}{s + jk_0\sqrt{\epsilon_1}}. \tag{43}$$

The final term on each side of (43) is added to remove the pole on the right side of (43) at $s = -j\sqrt{\epsilon_1}\,k_0$. The left side of (43) is analytic for Real $(s) > -$Real (jk); therefore, the left side of (43) is the analytic continuation of the right side, and both sides may be equated to some function $H(s)$.

Using the edge conditions, the left side of (43) is asymptotic to $1/|s|$ as $s \to \infty$ in the right half plane, and the right side is asymptotic to a constant as $|s| \to \infty$ in the left half plane. By Liouville's theorem, $H(s)$ is zero;

$$R_1(s) - R_2(s) = 4k_0\epsilon_1^{-1/2}\epsilon\cosh\frac{\epsilon_2}{\sqrt{\epsilon_1}}k_0aM_-(-jk_0\sqrt{\epsilon_1})M_+(s). \tag{44}$$

Solving (37) in a similar manner, one finds

$$R_1(s) + R_2(s) = \frac{-4jk_0\epsilon_1^{-1/2}\epsilon\sinh\dfrac{\epsilon_2}{\sqrt{\epsilon_1}}k_0aN_-(-j\sqrt{\epsilon_1}\,k_0)N_+(s)}{\sqrt{k + \sqrt{\epsilon_1}\,k_0}\sqrt{k - js}} \tag{45}$$

where

$$e^{-jWa}\cos Wa = N_+(s)N_-(s) \tag{46}$$

$$N_+(s) = [\cos ka]^{1/2}\exp\left[-\chi_2(s) - \frac{jaW}{\pi}\cos^{-1}\frac{s}{jk}\right] \cdot \prod_{n=1}^{\infty}\left(1 + \frac{s}{P_n}\right)e^{-s/P_n} \tag{47}$$

$$\chi_2(s) = s\sum_{n=1}^{\infty}\left[\frac{a}{(n - \frac{1}{2})\pi} - \frac{1}{P_n}\right] + \frac{as}{\pi}\left[\ln\frac{\pi}{2jka} + 1 - C'\right] \tag{48}$$

$$P_n = \sqrt{\frac{(n - \frac{1}{2})^2\pi^2}{a^2} - k^2} \qquad n = 1, 2, 3, \cdots. \tag{49}$$

To obtain $R_1(s)$ and $R_2(s)$, add and subtract (44) and (45) with the result that

$$\left.\begin{matrix} R_1(s) \\ R_2(s) \end{matrix}\right\} = 2k_0\epsilon_1^{-1/2}\epsilon\left[\pm\cosh\frac{\epsilon_2}{\sqrt{\epsilon_1}}k_0aM_-(-jk_0\sqrt{\epsilon_1})M_+(s) - \frac{j\sinh\dfrac{\epsilon_2}{\sqrt{\epsilon_1}}k_0aN_-(-j\sqrt{\epsilon_1}\,k_0)N_+(s)}{\sqrt{k + \sqrt{\epsilon_1}\,k_0}\sqrt{k - js}}\right] \tag{50}$$

where the $+$ and $-$ signs are used with $R_1(s)$ and $R_2(s)$, respectively.

The Determination of the Scattered Field

In the regions $x>a$, $x<-a$, the scattered field is given by the inverse transforms

$$H_y(x,z)=\frac{1}{2\pi j}\int_{c_0-j\infty}^{c_0+j\infty}\begin{Bmatrix}A(s)\\D(s)\end{Bmatrix}\cdot\exp\left[-jW(|x|-a)+sz\right]ds,\quad x\gtrless a. \qquad (51)$$

Using (32), (33), and (50), (51) becomes

$$H_y(x,z)=\frac{k_0\epsilon}{\epsilon_1^{1/2}\pi j}\int_{c_0-j\infty}^{c_0+j\infty}\cdot\left[\pm\cosh\frac{\epsilon_2}{\sqrt{\epsilon_1}}k_0aM_-(-j\sqrt{\epsilon_1}k_0)M_+(s)-\frac{j\sinh\frac{\epsilon_2}{\sqrt{\epsilon_1}}k_0aN_-(-j\sqrt{\epsilon_1}k_0)N_+(s)}{\sqrt{(k+\sqrt{\epsilon_1}k_0)(k-js)}}\right]\cdot\frac{e^{-jW(|x|-a)+sz}ds}{\epsilon_2 s\pm\epsilon_1 W} \qquad (52)$$

in which the upper and lower signs pertain to $x>a$ and $x<-a$. Contours for evaluating (52) are sketched in Fig. 3.

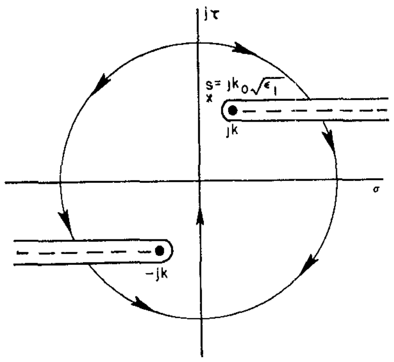

Fig. 3—Contour for evaluating (52), $k>0$, $\epsilon_1>0$, $\epsilon_2>0$.

If $x>a$, $\epsilon_2>0$, there is a pole in the right half plane, and if $x<-a$, $\epsilon_2<0$, there is a pole in the right half plane. The residues at these poles yield the surface wave H_{ys} which travels in the negative z direction.

$$H_{ys}=\frac{2k_0|\epsilon_2|}{\epsilon_1^{1/2}}\left[\cosh\frac{\epsilon_2}{\sqrt{\epsilon_1}}k_0a\overline{M_+(j\sqrt{\epsilon_1}k_0)}^2-\frac{j\sinh\frac{\epsilon_2}{\sqrt{\epsilon_1}}k_0a\overline{N_+(j\sqrt{\epsilon_1}k_0)}^2}{k+\sqrt{\epsilon_1}k_0}\right]\cdot\exp\left[-\frac{k_0|\epsilon_2|(|x|-a)}{\sqrt{\epsilon_1}}+j\sqrt{\epsilon_1}k_0z\right]. \qquad (53)$$

Note that because of the unidirectional character of the surface wave, there is a surface wave on only one of the two conductors at any given time. No surface wave poles exist for $x>a$, $\epsilon_2<0$, and $x<-a$, $\epsilon_2<0$, because the residues of the apparent poles at $s=-jk_0\sqrt{\epsilon_1}$ vanish.

The radiation field is found by evaluating (52) by the method of steepest descents. Set

$$z=r\cos\theta \qquad (54)$$

$$\pm x-a=r\sin\theta,\qquad x\gtrless a. \qquad (55)$$

Then for $kr\gg 1$, one may easily show that the radiation field $H_y(r,\theta)$ is

$$H_y(r,\theta)=\frac{2k_0\epsilon e^{-j(kr-\pi/4)}}{\sqrt{\pi k\epsilon_1 r}\left[\epsilon_1-j\epsilon_2\cot|\theta|\right]}\cdot\left[\cosh\frac{\epsilon_2}{\sqrt{\epsilon_1}}k_0aM_-(-jk_0\sqrt{\epsilon_1})M_+(-jk\cos\theta)-\frac{j\sinh\frac{\epsilon_2}{\sqrt{\epsilon_1}}k_0aN_-(-j\sqrt{\epsilon_1}k_0)N_+(-jk\cos\theta)}{\sqrt{2k}(k+\sqrt{\epsilon_1}k_0)\sin(|\theta|/2)}\right] \qquad (56)$$

which is symmetrical with respect to the z-axis ($\theta=0$).

In the region $a>x>-a$ the scattered field is represented by the inverse transform

$$H_y(x,z)=\frac{1}{2\pi j}\int_{c_0-j\infty}^{c_0+j\infty}\left[B(s)e^{-jWx}+C(s)e^{jWx}\right]e^{sz}ds \qquad (57)$$

where

$$B(s)=\frac{-R_1(s)e^{-jWa}+R_2(s)e^{jWa}}{2j\sin 2Wa(\epsilon_2 s+\epsilon_1 W)} \qquad (58)$$

$$C(s)=\frac{R_1(s)e^{jWa}-R_2(s)e^{-jWa}}{2j\sin 2Wa(\epsilon_2 s-\epsilon_1 W)}\cdot \qquad (59)$$

The quantity $C(s)$ has a pole at $s=-j\sqrt{\epsilon_1}k_0$, $\epsilon_2>0$; the residue at this pole is equal to minus one so that the incident wave is canceled in the region $a>x>-a$, $z>0$. The residue of $B(s)$ at $s=j\sqrt{\epsilon_1}k_0$, $\epsilon_2>0$ yields the dominant reflected wave H_{y0}.

$$H_{y0}=k_0\epsilon_1^{-1/2}|\epsilon_2|\left[\coth\frac{|\epsilon_2|}{\sqrt{\epsilon_2}}k_0a\overline{M_+(jk_0\sqrt{\epsilon_1})}^2+\frac{j\tanh\frac{|\epsilon_2|}{\sqrt{\epsilon_1}}k_0a\overline{N_+(jk_0\sqrt{\epsilon_1})}^2}{(k+\sqrt{\epsilon_1}k_0)}\right]\cdot\exp\left[-\frac{\epsilon_2}{\sqrt{\epsilon_1}}k_0x+j\sqrt{\epsilon_1}k_0z\right],\quad z<0. \qquad (60)$$

Although (60) was derived for $\epsilon_2>0$, one may also show that it is valid for $\epsilon_2<0$, and that the incident wave is canceled for $z>0$, $\epsilon_2<0$ when $a>x>-a$. Higher order reflected modes occur at the zeros of $\sin 2Wa$ which are in the right half plane.

TABLE I
SUMMARY OF PROPERTIES IN THE DIFFERENT FREQUENCY RANGES

Ω	ϵ_1	ϵ_2	k^2	Surface wave	Radiation field	Remarks
$0<\Omega<\Omega_1$	+	−	−	Lower conductor	No	
$\Omega_1<\Omega<R$	+	−	+	Lower conductor	Yes	
$R<\Omega<\Omega_2$	−	+	+	Evanescent		Not treated, incident wave evanescent
$\Omega_2<\Omega<\Omega_3$	+	+	−	Upper conductor	No	
$\Omega_3<\Omega<\infty$	+	+	+	Upper conductor	Yes	

RESULTS IN THE DIFFERENT FREQUENCY RANGES

As previously mentioned, we have assumed that $\epsilon_1>0$ so that both the surface wave and the TEM wave in the parallel-plane waveguide are propagating. We shall exclude the frequency region $R<\Omega<\sqrt{1+R^2}$ from further consideration, because here $\epsilon_1<0$ and these waves are evanescent. Eq. (5) shows that $\epsilon_2<0$ for $\Omega<R$ and $\epsilon_2>0$ for $\Omega>R$, if the static magnetic field is in the positive y direction. Reversing the direction of the static magnetic field changes the sign of ϵ_2.

We have also assumed that k^2 is real and positive. It is known that $k^2>0$, if $\Omega_1<\Omega<\Omega_2$, or $\Omega_3<\Omega<\infty$ [5]. Here

$$\Omega_1 = -\frac{R}{2} + \sqrt{\frac{R^2}{4}+1} \tag{61}$$

$$\Omega_2 = \sqrt{1+R^2} \tag{62}$$

$$\Omega_3 = \frac{R}{2} + \sqrt{\frac{R^2}{4}+1}. \tag{63}$$

In the other frequency ranges $0<\Omega<\Omega_1$ and $\Omega_2<\Omega<\Omega_3$, $k^2<0$ and in the solution k must be replaced by jk. In this case the space wave is exponentially damped and no power is radiated. Table I summarizes the behavior of the parallel-plane waveguide immersed in a lossless plasma with the static magnetic field in the positive y direction.

It is also instructive to consider the cases where the separation between the planes in very small or very large; i.e., the cases $a\to 0$ and $a\to\infty$. As $a\to 0$ the reflected power in the guide approaches 100 per cent, and the surface wave and radiation field vanish. When a is very large, the incident energy is concentrated in the vicinity of only one of the planes, and to a good approximation the scattered field is composed of two parts. The first portion of the field is obtained by using Seshadri's results for the scattering of a surface wave by a single semi-infinite, perfectly conducting plane in a plasma [5]. The second portion is the diffracted field of the second plane excited by a line source located at the edge of the first plane where most of the incident energy is concentrated.

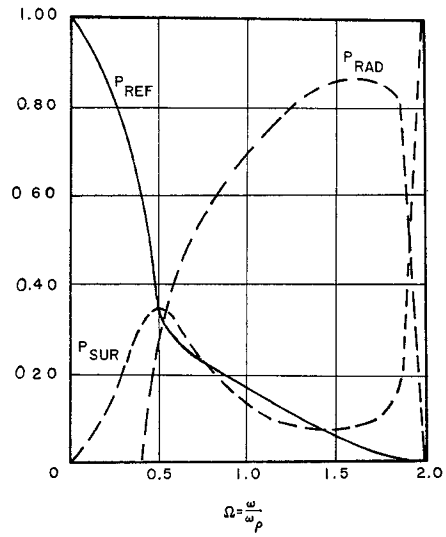

Fig. 4—P_{REF}, P_{RAD}, and P_{SUR} vs. Ω, $k_0a=\Omega\pi/4$, $R=2$.

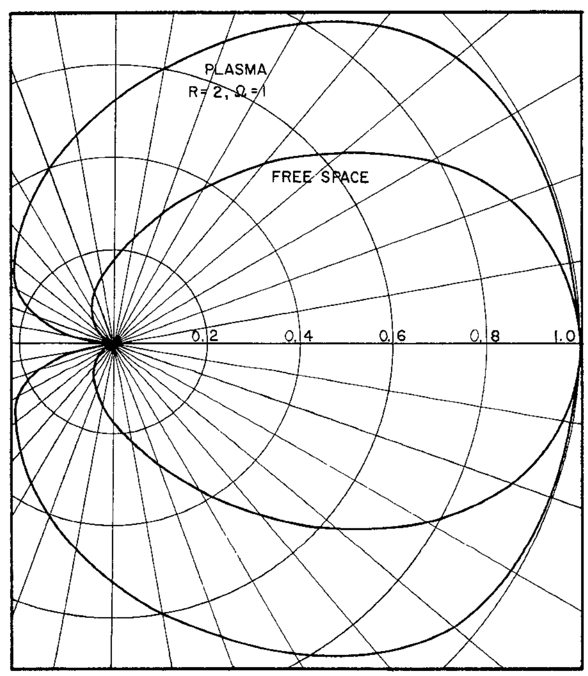

Fig. 5—Normalized radiation patterns, $k_0a=\pi/4$, $\Omega=1$.

Numerical Results

In Fig. 4, the fraction of the incident power reflected P_{REF}, radiated P_{RAD}, and launched in the surface wave P_{SUR}, are plotted in the frequency range $0<\Omega<2$ for $R=2$. For these calculations, a was arbitrarily chosen so that $k_0a=\Omega\pi/4$; *i.e.*, at the gyromagnetic frequency the spacing of the planes is one half a free space wavelength. Below $\Omega_1=0.414$, there is no radiation field, because the propagation constant k is imaginary. Above Ω_1 the percentage of power in the radiation field increases until a little below the gyromagnetic frequency. As one approaches the gyromagnetic frequency, ϵ_1 becomes very large and the incident field in the parallel-plane waveguide decays rapidly from its maximum at the lower conducting plane. The upper conducting plane then has a negligible effect, and nearly all the incident power is launched in the surface wave as is the case for a surface wave on a single semi-infinite, perfectly conducting plane [5].

Fig. 5 contains the normalized radiation pattern of power density for the parameters $R=2$, $\Omega=1$, and $k_0a=\pi/4$; the free space radiation pattern is plotted on the same scale for comparison. The plasma pattern is clearly much broader than the free space pattern.

Summary

The Wiener-Hopf technique yields the radiation field of a parallel-plane waveguide embedded in a lossless anisotropic plasma whose gyrotropic axis is perpendicular to the direction of propagation and parallel to the surface of the conductors. Expressions for the reflected TEM mode in the waveguide, the surface wave, and the radiation field in the far zone, are presented together with numerical results. Differences due to frequency changes are also discussed.

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